

Comparison of The Markowitz and Single Index Model in Optimal Portfolio

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Abstract: In this report, we choose the data of twenty years the historical closing price of ten stocks, and the SPX index to illustrate the comparison between the results of the Markowitz Model and the Index Model by finding optimization inputs for different models. When the models are used, five different combination weights are displayed by adding five different constraints of the real-world situations. Two perspectives of view, minimizing risk and maximizing sharp ratio, tend to determine the stock weight of optimal portfolio applying for both models, respectively. Then generate a comparison based on the SPX index and ten stocks' historical prices in the past twenty years. The Index Model performs better as it has a better sharp ratio and higher return, and two constraints perform exactly the same while one constraint has a very difficult result compared to other constraints.

1. Introduction

The construction of an investment portfolio should become a normal state of fund investment. There is no specific time period and specific investment objects for the construction of the portfolio. For every investor, building an investment portfolio is essential, and it is also a basic principle for long-term unremitting adherence. Only in this way can we "do not put eggs in the same basket"[1] and reduce the risk of fund investment. Investors choose higher returns to lower returns for a given level of risk, an optimal portfolio associated with recognized weight is under needs. Most investors mainly care about risk and expected return so that they should invest in stocks diversely. The purpose also as the first step to start is to maximize yield at a given risk level [2] or minimize risk at a given level of return. Both ideas produce an efficient portfolio and similar results to some extent. When there are no constraints, the model generates a global minimum point and it is where we actually start and find the efficient frontiers for each of five constraints. A set of frontiers gives an optimal portfolio. The comparison comes by applying different constraints and the nuance of the result of different models using. The most objective difference is that the combination of each constraint differs from each other substantially. The tangent point between efficient frontier and capital allocation line is the "best" combination showing weights allocated to each stock. For Markowitz Model, the number of estimations of yield is n , same for variance estimation; the number of covariances is $n(n-1)/2$. For Index Model, there are an estimated returns, beta, and risk estimation, plus 1 market factor.

In real life, we are eager to know how the chosen constraints influence the "best" result due to the historical data. This comparing is based on expected return and risk by application of two models.

The remainder of the article is organized as follows. Section 2 introduces the data collection. Section 3 presents the method used. In Section 4, the article shows the analysis of the results. Finally, the conclusion is presented in the last Section.

2. Data

In this research, ten stocks ranging from technology, finance, and travel industries were elected to conduct the entire research modeling, the SPX index data were also used as a reference. The raw

stock data came from the 20-year daily stock, which was extracted from the Bloomberg Machine. This chapter will firstly describe each company information basis and its stock information, then it will show the steps of the raw data processing.

2.1 Company basis and stocks

2.1.1 Adobe Inc. (ADBE)

Adobe is a multinational computer software company founded in 1982 and now headquartered in San Jose California. It is specialized in software for the creation and publication of digital content, such as photography, animation, and video. Nowadays, its flagship software such as Adobe Photoshop for photo editing and Portable Document Format (PDF) is serving millions of users worldwide. In 2021, Adobe expanded into digital marketing software and was recognized as one of the top global leaders in its field. As displayed in Figure 1, Adobe's price was around 35 USD between the first 15 years and jumped to over 450 USD in 2021.

2.1.2 International Business Machines Corporation (IBM)

IBM is an American multinational technology organization headquartered in New York City; it operates business in over 170 countries worldwide. It manufactures and sells computer components ranging from software and hardware, and it also provides hosting and consulting services related to technologies. According to Figure 1, IBM's stock price was remained steadily between 2001 to 2009, around 100 USD per share, and got doubled in the later 12 years.

2.1.3 SAP SE (SAP)

SAP SE is a German software corporation headquartered in Walldorf. It is well known for its business operation management and customer relation software, also called ERP. Meanwhile, it generates the highest revenue amount those non-American software companies and is one of the largest German firms by market capitalization. Its stock price has increased gradually in the last 20 years, from 36.52 USD in 2001 to 174.06 in 2021.

2.1.4 Bank of America Corporation (BAC)

The Bank of America provides both investment and basic financial services for its client. It was founded in San Francisco but is now currently headquartered in Charlotte, North Carolina. It is well known as the second-largest banking institution in the US as well as the eighth largest bank in the world. Its stock price fluctuates between around 20 USD to 60 USD in the last 20 years.

2.1.5 Citigroup Inc. (C)

Citigroup is a financial service headquartered in New York City, it was initially formed by the merger of Citicorp and Travelers Group in 1998. As the third-largest banking institution in the US, it operates through various segments, such as Global Consumer Banking; Institutional Clients Group. As a holding company. As illustrated by Figure 1, its stock price was initially climbed from 450 USD in 2001 to around 650 USD in 2007, then dropped to around 50 USD in 2008, then gradually increased to 100 USD in 2021.

2.1.6 Wells Fargo & Company (WFC)

Wells Fargo & Company is an American multinational financial service company with headquarter in San Francisco. It operates in 35 countries with over 70 million existing customers. Its stock price was around 35 USD before 2012 and increased to around 80 USD in 2021.

2.1.7 The Travelers Companies, Inc. (TRV)

The Travelers is a US insurance company that incorporates in Minnesota but headquarter in New York City. It is the second-largest writer of U.S. commercial property-casualty insurance and the sixth largest of U.S. personal insurance. It remained in the Dow Jones Industrial Average since 2008. The stock price increases steadily each year from 46.55 in 2001 to 256.03 USD in 2021.

2.1.8 Southwest Airlines Co. (LUV)

Southwest is one of the major US airlines and the world’s largest low-cost carrier. The airline was headquartered in Dallas and operates service to 121 destinations in the US and ten additional foreign countries. Currently, it has 60,000 employees with 4,000 flights during peak season. The stock price was around 15 USD in 2001 and drop to less than 10 USD in 2009, then increased to around 60 USD in 2021.

2.1.9 Alaska Air Group, Inc (ALK)

Alaska Air Group is an airline holding company in SeaTac. It operates with two certificated airlines, Alaska Airlines and Horizon Air. In the last 20 years, the stock price was increased dramatically from 7.12 USD in 2001 to 71.99 USD in 2021.

2.1.10 Hawaiian Holdings, Inc. (HA)

Table 1. the description of each company

	Annual Average Return	Annual StDev	beta	alpha	Residual Stdev
SPX	7.54%	14.85%	1.00	0.00	0.00%
ADBE	19.58%	31.79%	1.42	0.09	23.75%
IBM	4.75%	23.18%	1.01	-0.03	17.63%
SAP	12.00%	33.91%	1.48	0.01	25.78%
BAC	11.10%	39.34%	1.60	-0.01	31.40%
C	1.03%	42.47%	2.01	-0.14	30.26%
WFC	8.89%	28.13%	1.05	0.01	23.40%
TRV	9.06%	19.96%	0.80	0.03	16.00%
LUV	9.85%	31.80%	1.15	0.01	26.82%
ALK	17.43%	37.73%	1.18	0.09	33.43%
HA	26.87%	62.07%	1.63	0.15	57.16%

Hawaiian Holding, Inc. provides service amount the island of Hawaii between Hawaii and several West Coast Gateway cities and destinations in the South Pacific. It is the main commercial flight operator between Hawaii and US, the headquarter is in Honolulu. The stock price increased from 3.09 in 2001 and peaked at 57 USD in 2017 then dropped back to 23.16 USD in 2021.

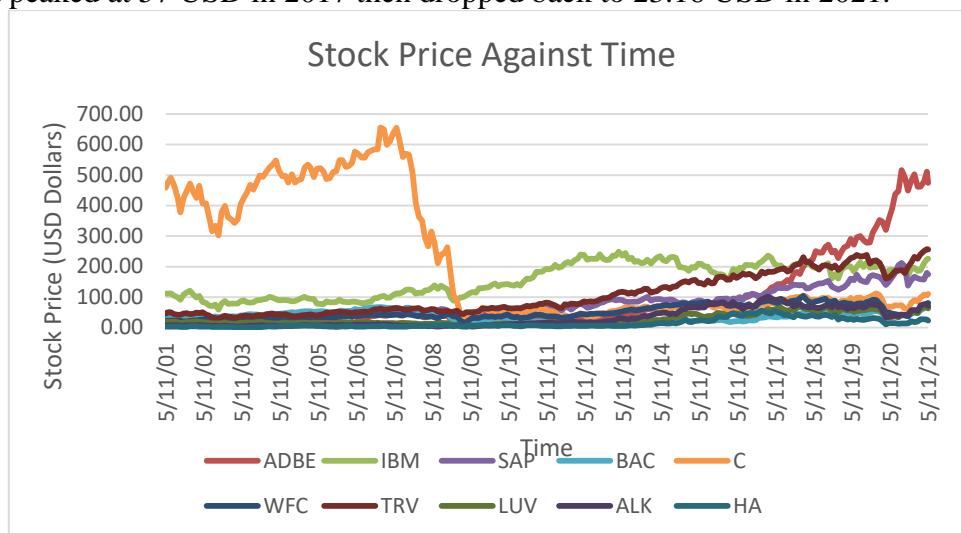


Figure 1. Portfolio stocks price against time.

2.2 Data Processing

The raw data was on a daily basis, by applying the algorithm in Excel, the average of each 5 days stock data was calculated to form the weekly data, and then every 4 weeks were used to form the monthly average stock prices. Since data is more accurate, monthly data can flat out the variations and pick up seasonality more reliably.

Equation (1) below was used to calculate the return of each stock for monthly data, where R_i indicates the return, P_1 is the current stock price and P_0 is the stock price of the previous month.

$$R_i = \frac{P_1}{P_0} - 1 \quad (1)$$

After finding the returns, relative data will be calculated for each stock, they are Annual Average Return, Annual Standard Deviation, Beta and Alpha, and Residual Returns and Standard Deviation.

3. Method

Given the explanation of the basic formula that we use to generate the optimal portfolio selection problem, used for both the Markowitz and Index model.

Notations

The set of instruments' average returns: $\vec{\mu} = \{\mu_1, \mu_2, \dots, \mu_n\}^T$

The unknown set of instruments' weights: $\vec{\omega} = \{\omega_1, \omega_2, \dots, \omega_n\}^T$;

The set of instruments' standard deviations: $\vec{\sigma} = \{\sigma_1, \sigma_2, \dots, \sigma_n\}^T$;

The set of instruments' betas: $\vec{\beta} = \{\beta_1, \beta_2, \dots, \beta_n\}^T$;

The set of the residuals' standard deviations: $\{\sigma(\varepsilon_1), \sigma(\varepsilon_2), \dots, \sigma(\varepsilon_n)\}^T$;

An auxiliary vector: $\vec{v} = \{\omega_1\sigma_1, \omega_2\sigma_2, \dots, \omega_n\sigma_n\}^T$;

The matrix of instruments' cross-correlation coefficients:

$$P = \begin{pmatrix} \rho_{11} & \cdots & \rho_{1n} \\ \vdots & \ddots & \vdots \\ \rho_{n1} & \cdots & \rho_{nn} \end{pmatrix} \quad (2)$$

3.1 Index Model

The model has been developed by William Sharpe in 1963[3], mathematically, The formula for the Markowitz Model portfolio return:

$$RP = \vec{\omega} \cdot \vec{\mu}^T \quad (3)$$

The formula for the Markowitz Model portfolio standard deviation:

$$\sigma_P = \sqrt{(\sigma_M \beta_P)^2 + \sum_{i=1}^n \omega_i^2 \sigma^2(\varepsilon_i)} \quad (4)$$

Where $\beta_P = \vec{\omega} \cdot \vec{\beta}^T$

3.2 Markowitz Model

The basic formula is originally introduced by Harry Markowitz (1952, 1959) who derived the expected rate of return and an expected risk measure for a portfolio of assets. [4]

The formula for the Index Model portfolio return:

$$RP = \vec{\omega} \cdot \vec{\mu}^T \quad (5)$$

The formula for the Index Model portfolio standard deviation:

$$\Sigma p = \sqrt{\vec{v}P\vec{v}^T} \quad (6)$$

3.3 Model constraints

In addition to the two portfolio optimizations models, five optimizations' constraints were also applied to both models with respect to simulating some real-life situations. This section will explain the 5 constraints.

3.3.1 constraint 1

constraint 1 is used to simulate regulation T by Financial Industry Regulatory (FINRA)[5]. It indicates that firms can only lend customers up to 50% for margin security for the new or initial purchase price. As referred to in Equation (2), the absolute value of the sum of whole stock purchase should be less than 200% percent. Where w_i indicates the proportional amount of each stock purchase. cates the proportion amount of each stock purchase.

$$\sum_{i=1}^{11} |w_i| \leq 2 \quad (7)$$

3.3.2 Constraint 2

constraint 2 can be expressed in Equation (3), this is designed to simulate some arbitrary "box" constraints on weights. This shows the optimized portfolio when each stock weights are less than 100% percent [6].

$$|w_i| \leq 1, \text{ for } \forall i \quad (8)$$

3.3.3 constraint 3

constraint 3 does not have any additional optimization constraints. This is used to observe how the optimized portfolio performs in general if it does not have any constraints [7].

3.3.4 constraint 4

constraint 4 is designed to simulate the current restriction for the mutual fund in the US. Which is a US open-ended mutual fund is not allowed to have any short positions [8], referred by the Investment Company Act of 1940, Section 12(a)(3). The math formula can be expressed as Equation (4).

$$w_i \geq 0, \text{ for } \forall i \quad (9)$$

3.3.5 constraint 5

constraint 5 removes the SPX index from the portfolio. In this way, we can determine whether the index has a positive or negative impact on the whole portfolio performance [9]. The mathematical expression can be shown as Equation (5), where w_1 indicates the first item (SPX index) in the portfolio.

$$w_1 = 0 \quad (10)$$

4. Result Analysis

The processed data from Table 1 was used to conduct the minimal variance portfolio of two model optimization methods under several constraints individually. This section will firstly analyze the result of the Index model and then the Markowitz model.

4.1 Analysis of Index Model

The optimized weights versus constraints under the index model can be found in Table 2. It can be observed that constraint 2 and constraint 3 have the distribution of the exact weight. This indicates that under the "free" model, none of the weights has weights over 100%. constraint 4 has a very different weight distribution compared to others, this indicates that it eliminates all the weights that have a negative value while the rest have similar weights compared to other portfolios.

Table 3 shows the return and sharp ratio comparison under the Index model. In general, most of these portfolios have a return just below 20% with a sharp ratio of around 0.9. However, constraint 4 performs does not perform as well as others, with the lower return of 17.28% and the sharp ratio of 0.78 separately.

Table 2. Stock weights table under Index model.

	Constr 1	Constr 2	Constr 3	Constr 4	Constr 5
SPX	48.2%	54.94%	54.94%	0.00%	0.00%
ADOBE	35.9%	38.26%	38.26%	40.84%	52.54%
IBM	-14.5%	-22.71%	-22.71%	0.00%	-20.56%
SAP	2.0%	2.98%	2.98%	0.00%	8.64%
BAC	-0.6%	-2.31%	-2.31%	0.00%	0.79%
C	-34.8%	-37.58%	-37.58%	0.00%	-41.18%
WFC	4.0%	4.25%	4.25%	0.00%	9.50%
TRAVEL	28.4%	28.63%	28.63%	26.27%	42.11%
LUV	3.6%	4.00%	4.00%	0.00%	8.47%
ALK	17.7%	18.66%	18.66%	20.52%	25.23%
HA	10.2%	10.89%	10.89%	12.38%	14.46%

Table 3. Return and Sharp Ratio table under Index model.

	Return	Sharp Ratio
constraint 1	18.9%	0.898
constraint 2	19.81%	0.901
constraint 3	19.81%	0.901
constraint 4	17.28%	0.740
constraint 5	23.79%	0.892

Table 4. Stock weights table under Markowitz model.

	Constr 1	Constr 2	Constr 3	Constr 4	Constr 5
SPX	39.2%	50.09%	50.09%	0.00%	0.00%
ADOBE	28.2%	35.81%	35.81%	50.27%	50.82%
IBM	-0.1%	-22.32%	-22.32%	0.00%	-19.51%
SAP	0.5%	7.19%	7.19%	0.00%	13.81%
BAC	16.7%	29.74%	29.74%	0.00%	36.74%
C	-48.1%	-66.14%	-66.14%	0.00%	-76.96%
WFC	19.8%	20.56%	20.56%	0.00%	28.73%
TRAVEL	30.2%	35.57%	35.57%	17.56%	49.95%
LUV	-1.8%	-14.83%	-14.83%	0.00%	-16.54%
ALK	7.7%	13.45%	13.45%	19.52%	18.45%
HA	7.6%	10.87%	10.87%	12.66%	14.50%

4.2 Analysis of Markowitz Model

Table 4 indicates the optimized portfolios under Markowitz Models. Similar to the Index model, constraints 2 and 3 have the exact same value. However, the weight distribution for constraint 1 is

very different compare to the weights for the Index model, as it invests less on the SPX with only 39.2% compared to 48.2% in the Index model.

As illustrated in Table 5, the return under most constraints is over 20%, where constraints 4 and 5 have a slightly higher value, around 25% each. Most of them have a sharp ratio around 1, except constraint 4, only 0.737.

Table 5. Return and max sharp ratio under against constraints.

	Return	SharpRatio
constraint 1	17.6%	0.994
constraint 2	21.33%	1.035
constraint 3	21.33%	1.035
constraint 4	24.75%	0.737
constraint 5	25.98%	1.021

4.3 Summary

In summary, both Markowitz and Index models have similar distributions for all the portfolio weights, return, and sharp ratios. While under some constraints, the distribution might be different, such as constraint 1.

Generally, the Markowitz model has a higher return, for most constraints around 7% for return and 12.9% for sharp ratio compared to the index model. This indicates under the same stocks, using the Markowitz model could get better performance.

Table 6. Return and sharp ratio error rate between two models.

	Ret Err	Sharp Err
constraint 1	-7.37%	9.62%
constraint 2	7.12%	12.90%
constraint 3	7.12%	12.90%
constraint 4	30.19%	-0.42%
constraint 5	8.44%	12.67%

5. Conclusion AND Reflection

Generally, the research went well, we were able to produce the required data for all the attributes. Specially, we both horizontally and vertically compared experiment results on different models and constraints.

In the comparison of the two models, the Markowitz model generally has a better performance as it has a higher return and sharp ratio [10]. This means, when a portfolio is calculated under both models, the first one would generally generate a higher return. However, to successfully produce a Markowitz model, it relies on the history data. In addition, the calculation complexity will also increase exponentially as the stock size increases.

When comparing the 5 different constraints, we could see that most of the indexes perform similarly on both models. Some constraints perform the same as the “free” constraint, and this still might largely depend on the stock selection and the property of the constraint. In this portfolio selection, only one of the constraints performed differently compared to others. Hence, we can conclude that in our portfolio selection, having a short position will generally give a better profit.

In this research, we have done many modelings and produced reasonable results for the portfolio. Nevertheless, there are some defects.

Firstly, we only considered ten stocks for this experiment, which was fairly small compared to the real portfolio size. Also, the stocks only come from three entirely different sectors, it cannot represent the whole image of the stock market or the performance of a specific field if we want to focus on some specific area. Hence, we could considerably improve our stock selections.

Next, we are using excel to conduct the modeling, this is very inefficient as excel can be very slow compared to the software. If we have more time; we would consider developing some specific software based on Python or C++ to improve the efficiency.

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